Optimal stopping in a principal-agent model with hidden information and no monetary transfers

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Abstract

We study optimal stopping rules in a simple principal agent framework when the exchange of contingent monetary transfers is infeasible. In each period, the agent privately learns the value from stopping today, but disagrees with the principal about when to stop. A stopping rule commits the principal to a stopping probability conditional on communication by the agent. The optimal stopping rule exhibits a finite deadline. Within the deadline, the agent can make one proposal to stop. In case of rejection, the relation is terminated. If the deadline is reached without a proposal, the agent obtains the decision right.

Keywords: Dynamic Mechanism Design, Optimal Stopping, Search, Delegation

JEL Codes:.

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1 Introduction

In many important situations of economic interest, decision makers face timing decisions, a.k.a. stopping problems, that involve the inter-temporal trade-off between taking an irreversible action today or, instead, forgoing current payoff and searching for a better opportunity tomorrow. This paper studies optimal stopping rules in a simple stopping problem when the decision maker lacks relevant information but can consult with an informed yet self–interested agent while the exchange of contingent monetary transfers is infeasible.

Stopping problems with hidden information are pervasive in practice. A leading example is the search for new employees where the employer has to decide whether to hire a current applicant or hope for a better candidate to arrive in the future. Recruiting, particularly for highly specialized jobs such as in academia, is routinely delegated to search committees who possess the superior ability required to assess applicants. For various reasons, conflict of interest may aries so that a committee member may not favor the candidate which is best for the organization as a whole, however. For example, she may perceive the best candidate as a competitor for promotion, may prefer candidates that benefit her own division, or due simply to personal chemistry. At the same time, it is rarely observed or may be institutionally impossible[1] to pay committee members based on the hiring outcome.

Similarly, essential strategic business decisions such as when to adopt new projects, make irreversible investments, launch a new product, enter a new market, or when to merge with another firm, to name a few, are stopping problems which headquarters/shareholders delegate to managers exactly because of their expertise. Moreover, contingent monetary transfers may not be feasible. For instance, while manager pay does depend on the overall success of the firm, it is notoriously difficult to fine-tune pay to the outcome of individual decisions (possibly because of the problem of incentivizing multiple tasks at the same time). Also, politicians face important timing decisions such as when to go to war, when to intervene in the economy, or when to push through one's own policy agenda and thereby have to rely on the advice of interested lobbyists without re-election concerns (the military, business representatives/unions, bureaucrats) who are not, or must not be, paid directly by the politician.

In this paper, we characterize optimal stopping rules for a simple principal agent stopping

[1] In German academia it is not possible to negotiate bonuses depending on one's hiring success.

problem where, in each period, the agent observes the current state of the world which determines the payoff from taking an irreversible action ("stopping") today. We consider the simplest setting in which the state evolves independently across time and can be either "high" or "low", corresponding to a high or low payoff from stopping. We adopt a mechanism design approach where, at the outset, the principal can commit to a dynamic stopping rule that specifies for each period the probability to stop depending on information communicated by the agent. In the benchmark case with publicly observable payoffs, a stopping rule specifies whether to stop or continue depending on the current state, and the optimal stopping rule trades off getting an immediate payoff with obtaining a delayed (discounted) higher payoff tomorrow. To capture conflict of interest between the parties, we assume that, while both parties agree about the direction of current payoffs, they have different preferences about when to stop: while the principal would ideally want to stop only when the current state is high, the agent would want to stop in both the high and the low state. Disagreement thus arises in the low state.

Our main result establishes the optimal stopping rule when payoffs are only privately observed by the agent. Under the optimal stopping rule the principal sets a deadline. Within the deadline, the agent can make exactly one proposal to stop. If he makes a proposal, it is accepted with some probability (which increases over time); with the opposite probability, the proposal is rejected and, in fact, no action is ever taken in the future so that stopping never occurs. If the agent does not make a proposal until the deadline, he is delegated the decision right immediately after the deadline (which is in our simple setup equivalent to stopping immediately after the deadline).

In designing the optimal stopping rule, the principal faces the problem that she seeks to implement a high stopping probability in the high and a low stopping probability in the low state, while, all else equal, the agent prefers a higher stopping probability in both states. How can the principal provide incentives for truth–telling in the low state and implement a small stopping probability at the same time?

One of the basic observations in this paper is that in the absence of monetary transfers, the only way to provide such incentives is to make the *future* stopping rule contingent on the agent's *current* communication. Doing so generates contingent continuation values for the agent which, at least partially, substitute for money as a screening instrument, because by promising a higher continuation value in return, the principal can induce the agent to truthfully announce the low payoff state even though this may imply a small probability to stop today.

All else equal, the principal can maximize the current incentives for truth–telling in the low state by promising the agent the highest possible continuation value when he announces the low state, and by threatening him with the lowest possible continuation value when he announces the high state. The design of continuation values is, however, associated with specific future costs for the principal. For example, supplying the agent with the highest possible continuation value requires the principal to delegate the decision right to the agent from tomorrow on, implying that stopping occurs tomorrow even if tomorrow's state is low. Reversely, supplying the agent with the lowest possible continuation value requires to never stop, even if the future state is high.

One of the key steps in our analysis is to determine the principal's costs from supplying the agent with a certain continuation value. We do so by setting up the principal's problem recursively. This allows us to determine jointly the principal's utility as a function of the agent's value and the optimal stopping rule.

The optimal stopping rule trades off the current benefits and future costs of providing truth-telling incentives. Under the optimal stopping rule, the agent is deterred to make a proposal to stop as long as he observes the low state in which case the next period is reached with probability 1. Hence, until the deadline, the stopping probability in the low state is kept at zero. On the other hand, the agent is induced to make a proposal to stop when he observes a high state for the first time. In this case, if the proposal is rejected, the relation terminates. The threat of termination provides the proper incentives for the agent to not make a proposal in the low state. Moreover, under the optimal stopping rule, the probability to accept a proposal increases with time. To make this incentive compatible, the continuation value after announcing the low state needs to gradually increase. Indeed, at the deadline, the continuation value after announcing the low state reaches its maximal possible value, which can only be supplied by delegating the decision to the agent.

Related Literature

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2 Model

We begin by describing our setting in stylized terms and subsequently offer a number of specific interpretations. We consider a principal (she) who faces a stopping problem where in each of infinitely many periods $t = 1, 2, \ldots$, she has to take an irreversible action ("stop") today, or, instead, continue and stop tomorrow. The principal has only incomplete information about the payoff of stopping today and can consult with an agent (he) who is better informed but has possibly different preferences about when to stop. It is commonly known that the payoff from stopping depends on a state s which can take on the two values L or H, where in each period, state H is drawn with probability p, independent of the states in the previous periods. If stopping occurs in state s, the principal obtains the payoff τ_s, and the agent obtains the payoff θ_s where $\tau_L < \tau_H$ and $\theta_L < \theta_H$. Let $\bar{\tau}$ and $\bar{\theta}$, respectively, be the expected payoff from stopping for the principal and the agent, respectively. The current payoff of continuing to the next period is normalized to zero for both parties, and both parties discount future payoffs by the common discount factor $\delta \in [0, 1)$.

While the principal does not observe the state, the agent learns the current (yet not the future) state at the beginning of each period. We also refer to the state as the agent's type. The main question of the paper is how the principal can optimally provide incentives for the agent to reveal his information. We study this question in a context in which contingent monetary transfers are infeasible and adopt a mechanism design approach where, at the outset, the principal can commit to a stopping rule that specifies the probability to stop or continue depending on communication by the agent.

We proceed with a selection of applications that fit this stylized setup:

• The principal represents an organization aiming to fill a vacancy. The agent represents a recruitment committee with superior competence to assess job candidates. Exactly one candidate can be hired (corresponding to "stop"), and in each period, one candidate is sampled who disappears from the market if not hired.

• The principal corresponds to a firm's headquarters who seek to adopt a new business project, make an irreversible investment, or launch a new product. The agent is a manager with better information about the profitability of these actions. "Stopping" corresponds to adopting the new project, making the investment, or launching the new product. Monetary transfers contingent on completing the task are infeasible because of the presence of other (unmodelled) tasks.

• The principal is an accountant of an organization overseeing a limited budget whose size is normalized to 1. The agent is a division who, in each period, may be allocated some money to fund its operations. The amount x spend in a given period corresponds to the probability to "stop", resulting in a utility for the principal and the agent of $\tau_s x$ and $\theta_s x$ in state s. The accountant does not care about the amount of money spent as long as it is within the limits of the budget.

• The principal is a politician who has to decide to go to war ("stop"). The agent is a hawkish general who can better assess the prospects of winning and is more eager to fight than the politician.

3 Symmetric information benchmark

In this section, we describe the principal's and the agent's optimal policy under symmetric information when also the principal can observe the state. Consider first the principal. The principle of optimality[2] implies that the optimal policy is stationary and can thus take three possible forms:

(a) stop in both states, inducing expected utility $\bar{\tau}$;

(b) stop in state H and continue in state L, inducing expected utility

$$w = p\tau_H + (1-p)\delta w \quad \Leftrightarrow \quad w = \frac{p\tau_H}{1-\delta(1-p)}; \tag{1}$$

(c) stop in neither state, inducing expected utility 0.

By comparing expected utilities, we obtain the following lemma:

Lemma 1. *Under symmetric information, it is optimal for the principal to:*

(a) stop in both states if and only if $\tau_L \geq \delta\bar{\tau}$;

(b) stop in state H and continue in state L if and only if $\tau_L < \delta\bar{\tau}$ and $\tau_H \geq 0$;

(c) stop in neither state if and only if $\tau_H < 0$.

The optimal policy of the agent is analogous with τ replaced by θ.

[2] The principle of optimality, due to Bellman, says that an optimal policy has the property that whatever the initial decision is, the remaining decisions must constitute an optimal policy in the future problem that results from the first decision.

Of course, if there is no conflict of interest, and the principal and the agent agree on the optimal policy, then the problem is trivial also with asymmetric information. To capture conflict of interest, we focus on the case that the principal would like to continue in state L and stop in state H, and that the agent would like to stop in both states.[3] By Lemma 1, we therefore assume that

$$\tau_L < \delta\bar{\tau}, \quad \tau_H > 0, \qquad \theta_L > \delta\bar{\theta}, \quad \theta_H > 0. \tag{2}$$

Before turning to the formal analysis, we outline the fundamental problem for the principal when there is asymmetric information. In this case, due to assumption (2), the principal cannot implement the policy which is optimal under symmetric information. Clearly, if the principal asked the agent to report her information and stopped in the (reported) state H and continued in the (reported) state L, then the agent would, in state L, lie and report state H.

How can the principal provide incentives for the agent to reveal his information? Absent monetary transfers, the available instruments to induce information revelation may appear rather limited. In fact, if there was only a single period, it would be impossible to elicit information at all from the agent, because given that both types prefer to stop, the agent would simply announce the report with the largest stopping probability, independent of his true type.

This changes when there are multiple periods. In fact, by making the *future* stopping rule contingent on the current report the principal can effectively design a contingent continuation value for the agent which, at least partially, can serve as a substitute for contingent monetary transfers in that it allows the principal to deter the agent from announcing the report with the higher stopping probability by promising a higher continuation value in return. We now develop these considerations formally and look for the principal's optimal mechanism.

[3]Conflict of interest arises also in the case in which the principal would like to stop in both states, and the agent would like to stop only in the high state. Observe that in this case private information does not add anything interesting, because the stopping rule which stops with probability 1 independently of the agent's report implements the principal's optimal policy also with private information.

4 Mechanisms with asymmetric information and recursive problem

Since the principal has commitment, the revelation principle implies that the optimal mechanism for the principal is in the class of direct and incentive compatible mechanisms. A direct mechanism requires the agent to announce in each period t a report $\hat{s} \in \{L, H\}$ about the state and specifies a stopping probability $x_{t\hat{s}}(h_t)$ depending on the agent's report $\hat{s}$ and on the history $h_t \in \{L, H\}^{t-1}$ of previous reports by the agent.[4] Unless endangering clarity, we will omit the dependency of $x_{t\hat{s}}(h_t)$ on t and h_t, and simply write $x_{\hat{s}}$ instead.

Two specific mechanisms will play an important role in the analysis below.

• We call the mechanism that stops with probability 1 in both states "full delegation" because it induces the same outcome as when the agent had the decision right.

• We call the mechanism that stops with probability 0 in any period "resignation" because it means that no action is ever taken.

We will now re–interpret the notion of a direct mechanism. Note that any mechanism induces a certain continuation value for the agent, which is the agent's expected utility in the next period conditional on continuing to the next period. Let $v_{\hat{s}}$ be the agent's continuation value when reporting $\hat{s}$ in the current period. The continuation value depends on the agent's current report since the stopping probabilities in the future periods depend on the current report. Since the state is drawn independently after any period, the agent's continuation value is, however, independent of the true state. Reversely, starting with a continuation value $v_{\hat{s}}$, there is a mechanism that generates this continuation value as long as

$$v_{\hat{s}} \in [0, \bar{\theta}]. \tag{3}$$

To see this, note the mechanism that is worst for the agent from tomorrow on is "resignation", yielding the agent a continuation value $v_{\hat{s}} = 0$. The mechanism that is best for the agent from tomorrow on is "full delegation", yielding the agent $v_{\hat{s}} = \bar{\theta}$ by (2). Because the agent's utility is linear in the stopping probabilities, the principal can generate any continuation value in $[0, \bar{\theta}]$ simply by randomizing between "resignation" and "full delegation".

[4] In some applications, x can be interpreted as a quantity. As indicated earlier, when stopping corresponds to making an irreversible investment, x may be seen as the size of the investment. Or, when the principal has a budget of size 1 to allocate to the agent over time, x may be seen as the amount of money allocated today.

Hence, the principal's problem of designing a set of history dependent stopping probabilities is equivalent to designing, for each report, the current stopping probability and the continuation value. Therefore, a direct mechanism amounts to a combination (x_H, x_L, v_H, v_L) with

$$0 \leq x_L, x_H \leq 1, \quad 0 \leq v_L, v_H \leq \bar{\theta}, \tag{4}$$

and the utility of agent type s when reporting $\hat{s}$ is

$$\theta_s x_{\hat{s}} + (1 - x_{\hat{s}})\delta v_{\hat{s}}. \tag{5}$$

This expression reveals the formal analogy between our dynamic problem and a standard static screening problem with monetary transfers. Our stopping probability corresponds to an "allocation" (quantity, probability of consumption, ...), and the continuation value for the agent plays a similar role as transfers in a standard screening problem. A key difference, however, is that continuation values are bounded between 0 and $\bar{\theta}$.

Given this notion of a direct mechanism, a mechanism is incentive compatible if it induces the agent to report his type truthfully:

$$\begin{array}{lrcl} IC_H: & x_H\theta_H + (1 - x_H)\delta v_H & \geq & x_L\theta_H + (1 - x_L)\delta v_L, \\ IC_L: & x_L\theta_L + (1 - x_L)\delta v_L & \geq & x_H\theta_L + (1 - x_H)\delta v_H. \end{array}$$

Analogously to the agent, we will define the principal's expected utility from a mechanism as the sum of current value and continuation value. This will give rise to a recursive formulation of the principal's problem. Let v denote the agent's expected utility from a mechanism before having observed the state:

$$\star: \qquad v = p[\theta_H x_H + (1 - x_H)\delta v_H] + (1 - p)[\theta_L x_L + (1 - x_L)\delta v_L].$$

We refer to v as the agent's value, and we denote by $W(v)$ the principal's expected utility from the best mechanism that yields the agent v. We refer to W as the principal's value function. By the principle of optimality[5], the value function satisfies the Bellmann equation

$$\begin{array}{rcl} W(v) & = & \max\limits_{x_H, x_L, v_H, v_L} p[\tau_H x_H + (1 - x_H)\delta W(v_H)] + (1 - p)[\tau_L x_L + (1 - x_L)\delta W(v_L)] \\ & & \quad s.t. \qquad (4), (IC_H), (IC_L), (\star). \end{array}$$

[5] See footnote 2.

The Bellmann equation captures the fact that to provide the agent's value v in the optimal way, the principal also needs to provide the agent's continuation values v_H and v_L optimally. For infinite time horizon, the principal's maximum utility associated with providing continuation values v_H and v_L is again given by the value function W.

In analogy to screening problems with money, we may interpret $W(v)$ as the principal's "valuation for money". A complication that arises in the current context is, however, that the principal's valuation for money is endogenous.

The main result of the paper is that the optimal mechanism takes the form of a "deadline mechanism": The principal sets the agent a deadline t^*. Within the deadline, the agent can make exactly one proposal to stop. If the agent makes a proposal, it is accepted with some probability (depending on the time of the proposal); with the opposite probability, the proposal is rejected and "resignation" occurs. If the agent does not make proposal until the deadline, then "full delegation" immediately after the deadline occurs.

The next section presents the arguments for our main result.

5 Analysis

Our approach to finding the optimal mechanism is to solve for the value function. The optimal mechanism is then the solution to the principal's problem for the value v that maximizes W. To find the principal's value function, we proceed inductively, and first solve the problem when there are only finitely many periods $t = 1, \ldots, T < \infty$.

For time horizon T, we denote by $W^T(v)$ the principal's value from the best mechanism that supplies the agent with a value v. Again, by the principle of optimality, we have

$$\begin{aligned} W^T(v) \quad &= \quad \max_{x_H, x_L, v_H, v_L} \quad p[\theta_H x_H + (1 - x_H)\delta W^{T-1}(v_H)] + (1-p)[\tau_L x_L + (1 - x_L)\delta W^{T-1}(v_L)] \\ & \qquad\qquad s.t. \qquad (4), (IC_H), (IC_L), (\star). \end{aligned}$$

As we have remarked earlier, if there is only a single period ($T = 1$), all that can be implemented is a probability to stop x, independent of the true type. Thus, the agent's value is $v = \bar{\theta}x$, and accordingly the principal's value is

$$W^1(v) = \frac{\bar{\tau}}{\bar{\theta}} v. \tag{6}$$

We turn to our first result which says that the infinite horizon value function is the limit of the T–period value functions:

Lemma 2. *The limit* $\lim_{T\to\infty} W^T(v)$ *exists and is equal to* $W(v)$*.*

The argument is as follows. Observe first that for each v, $W^T(v)$ is increasing in T. Because $W^T(v)$ is bounded, the limit therefore exists. To see that the limit is equal to $W(v)$, take a feasible infinite horizon mechanism M, and consider the T–period mechanism M^T which in the first $T-1$ periods coincides with M, and which, for each history of reports, gives the agent in period T exactly the same value as the corresponding value under M. (This can be attained by appropriately randomizing between "full delegation" and "resignation".) Therefore, the mechanism M^T is incentive compatible by construction. Moreover, the principal's payoff from M^T and M is different only from period T on. Due to discounting, the value difference therefore vanishes as T grows large. This shows that $W(v)$ cannot be strictly larger than the limit of $W^T(v)$.

While the lemma says that the finite time value function for large time horizon approximates the infinite time value function, our inductive procedure will not only yield an approximately optimal mechanism. In fact, it turns out that the maximum value in the infinite and finite horizon model coincide for sufficiently large finite horizon.

We proceed by constructing the finite horizon value function W^T for all T.

5.1 Finite time horizon

We derive the value function with the help of a number of auxiliary lemmata. We start with three fundamental properties of the value function.

Lemma 3. *(i)* W^T *is concave.*
(ii) $W^T(v) \leq \tau_H/\theta_H \cdot v$*;*
(iii) $\frac{W^T(v)-W^T(v')}{v-v'} \leq \tau_H/\theta_H$ *for all* $v, v', v \neq v'$*.*

To see concavity, consider a mechanism where after the agent's report, the principal randomizes and with some probability λ uses a mechanism that gives the agent v and herself $W^T(v)$ and with probability $1-\lambda$ uses a mechanism that gives the agent v' and herself $W^T(v')$. If the two mechanisms over which the principal randomizes are feasible, so is the new mechanism.

The new mechanism gives the principal $\lambda W^T(v) + (1-\lambda)W^T(v')$ and the agent $\lambda v + (1-\lambda)v'$. Hence, if the principal uses an optimal mechanism that generates this value for the agent, she is (weakly) better off. Hence, $W^T(\lambda v + (1-\lambda)v') \geq \lambda W^T(v) + (1-\lambda)W^T(v')$, establishing concavity.

To see the second property, observe that assumption (2) implies that in each state, the principal's payoff from stopping is at most the agent's payoff from stopping times the factor τ_H/θ_H. Hence, the principal can get at most the agent's value times this factor. Finally, part (iii) is an immediate consequence of parts (i) and (ii).

We next turn to the question which incentive constraint is binding at the optimum. Intuitively, recall that in the symmetric information benchmark case, the principal would optimally continue in state L and optimally stop in state H. If the principal offered this mechanism under asymmetric information, then because both agent types prefer to stop, agent type L would pretend to be type H. This suggests that the incentive constraint for type L is binding at the optimum. The next lemma confirms this intuition formally. Moreover, the lemma also establishes the familiar property that incentive compatibility implies monotone stopping probabilities, and this will allow us to ignore the incentive constraint for type H.

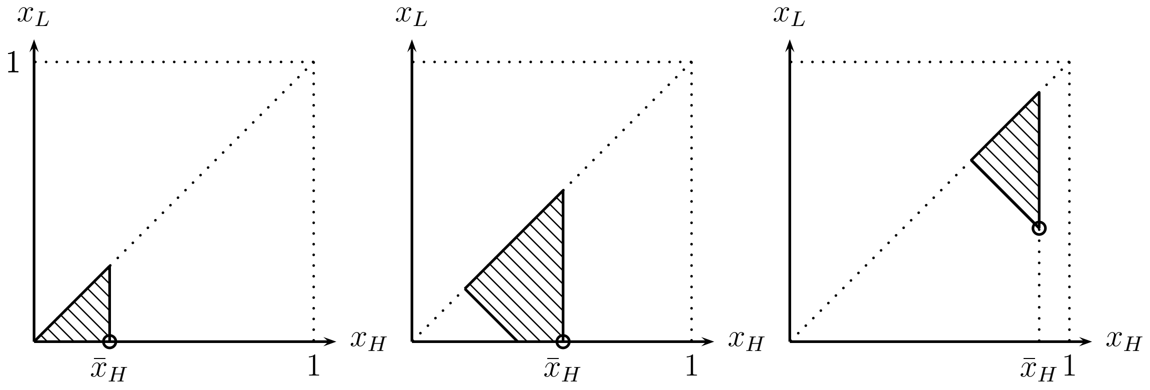
Lemma 4. *(i) A mechanism $(x_H^*, x_L^*, v_H^*, v_L^*)$ is optimal only if (IC_L) is binding and $x_H^* \geq x_L^*$. (ii) Moreover, if (IC_L) is binding and $x_H \geq x_L$, then (IC_H) is met.*

The lemma implies that we can replace the constraints (4), (IC_H), (IC_L), ($\star$) by the following set of constraints.[6]

$$
\begin{aligned}
IC_L^= : &\qquad x_L\theta_L + (1-x_L)\delta v_L = x_H\theta_L + (1-x_H)\delta v_H; \\
\tilde{\star} : &\qquad (\bar{\theta} - \delta v_H)x_H + \delta v_H = v; \\
M : &\qquad 0 \leq x_L \leq x_H \leq 1, \quad 0 \leq v_H, v_L \leq \bar{\theta}.
\end{aligned}
$$

The two equality constraints pin down the continuation values as a function of the stopping probabilities. Similarly as in a standard screening problem, this allows us to eliminate the continuation values from the objective so that the principal's problem becomes a maximization problem over the stopping probabilities alone. In fact, if we could choose continuation values arbitrarily, this would allow us to select an *arbitrary* pair (x_H, x_L) of (monotone) stopping

[6]To obtain ($\tilde{\star}$), we have inserted the binding (IC_L) in ($\star$).



$0 < v < \delta\bar{\theta}$ $\qquad$ $\delta\bar{\theta} < v < \delta\bar{\theta}^2/\theta_L$ $\qquad$ $\delta\bar{\theta}^2/\theta_L < v < \bar{\theta}$

Figure 1: Feasible stopping probabilities $X(v)$.

probabilities and then rest assured that we could find continuation values to make these stopping probabilities incentive compatible. However, because the continuation values are bounded in the current setup, this is not possible here. Instead, depending on the value v provided to the agent, only a subset of pairs of stopping probabilities can be made incentive compatible. The next lemma characterizes this subset of feasible stopping probabilities.

Lemma 5. *Let*

$$\underline{x}_H \equiv \max\left\{0, \frac{v - \delta\bar{\theta}}{\bar{\theta} - \delta\bar{\theta}}\right\}, \tag{7}$$

$$\bar{x}_H \equiv \frac{v}{\bar{\theta}}, \tag{8}$$

$$\xi_L(x_H) \equiv \frac{(\theta_L - \bar{\theta})x_H + v - \delta\bar{\theta}}{\theta_L - \delta\bar{\theta}}. \tag{9}$$

There are $v_H, v_L \in [0, \bar{\theta}]$ so that the mechanism (x_H, x_L, v_H, v_L) satisfies ($IC_L^=$), ($\tilde{\star}$), and (M) if and only if it holds:

$$(x_H, x_L) \in X(v) \equiv \{(x_H, x_L) \mid \quad \underline{x}_H \le x_H \le \bar{x}_H, \quad \max\{0, \xi_L(x_H)\} \le x_L \le x_H\}.$$

Moreover, v_H and v_L are pinned down by:

$$v_H(x_H) = \frac{v - \bar{\theta}x_H}{\delta(1 - x_H)}, \quad v_L(x_L, x_H) = \frac{(\theta_L - \bar{\theta})x_H - \theta_L x_L + v}{\delta(1 - x_L)}. \tag{10}$$

Figure 1 illustrate the set $X(v)$ graphically. Moving from left to right, the panels depict cases for increasing v. In the left panel, the agent's value is rather small. It is then feasible to implement zero stopping probabilities, because the agent's value can be generated by means of continuation values alone and with a zero current value. For $v = 0$, the triangle in the left panel becomes the point $(0, 0)$. In the middle panel, the agent's value is in an intermediate range so that it cannot be generated with continuation values alone, and the mechanism needs to stop at least in the high state with a positive probability. In the right panel, the agent's value is so large that it can be only supplied with positive stopping probabilities in both the high and low state. For $v = \theta$, the triangle in the right panel becomes the point $(1, 1)$.

An encircled point is the unique point in $X(v)$ which dominates all other points in the x_H and is dominated by all other points in the x_L dimension. As the next lemma will imply, such a point constitutes an optimal mechanism.

Lemma 6. *(i) At an optimal mechanism, $v_H = 0$.*
(ii) At an optimal mechanism, x_L is as small as possible.

It is easy to see from Lemma 5 that at a mechanism that has $v_H = 0$, the stopping probability in state H is maximal: $x_H = \bar{x}_H$. Together with part (ii), this implies that the corner point which is encircled in Figure 1 is optimal.

To see part (i) of Lemma 6, consider a mechanism that supplies the agent with v and has $v_H > 0$. Now modify the mechanism and set $v'_H = 0$ and $x'_H > x_H$ so that the utility of agent type H is unaffected (all other parts of the original mechanism remain unchanged):

$$\theta_H x'_H = \theta_H x_H + (1 - x_H)\delta v_H. \tag{11}$$

Clearly, the modification still supplies the agent with v. Moreover, it is easy to verify that the modification is still feasible. The modification evidently does not affect the principal's utility in state L. Conditional on state H, the original mechanism yields the principal expected utility

$$\tau_H x_H + (1 - x_H)\delta W(v_H) \leq \tau_H x_H + (1 - x_H)\delta \frac{\tau_H}{\theta_H} v_H = \tau_H x'_H, \tag{12}$$

where the inequality follows from part (ii) of Lemma 3, and the final equality from (11). But the right hand side is the principal's utility from the modified mechanism. Hence, the modification is an improvement.

Part (ii) of Lemma 6 is established in the appendix. Similarly to part (i), the idea is to construct a profitable modification by reducing x_L and increasing v_L, while maintaining the low type agent's utility. The difficulty is, however, that it is not always feasible to reduce x_L to 0 or increase v_L to $\bar{\theta}$. Therefore, one has to consider local changes and, in particular, we have to show that a local increase in v_L does not hurt the principal too much. To show this, we establish a bound on the rate at which W^{T-1} decreases by exploiting the concavity of the value function.

With Lemma 6 at hand, we can infer the optimal stopping probabilities and the corresponding continuation values by inspecting Lemma 5.

Lemma 7. *The optimal mechanism that gives the agent the value v is given by*

$$x_H^* = \frac{v}{\bar{\theta}}, \qquad x_L^* = \begin{cases} 0 & if \quad v \leq v_1 = \delta\bar{\theta}^2/\theta_L \\ \frac{\theta_L/\bar{\theta}\cdot v - \delta\bar{\theta}}{\theta_L - \delta\bar{\theta}} & if \quad v > v_1 \end{cases}, \tag{13}$$

$$v_H^* = 0, \qquad v_L^* = \begin{cases} \frac{\theta_L}{\delta\bar{\theta}} v & if \quad v \leq v_1 = \delta\bar{\theta}^2/\theta_L \\ \bar{\theta} & if \quad v > v_1 \end{cases}. \tag{14}$$

Lemma 7 exhibits two important features of the optimal mechanism which we will to argue that the value function is piece–wise linear. First, the optimal mechanism is qualitatively different, depending on whether v is larger or smaller than $v_1 = \delta\bar{\theta}^2/\theta_L$. Second, for each state s, either the stopping probability is linear in v and the continuation value v_s is constant, or the stopping probability is constant and the continuation value is linear in v.

For the case with two periods, since W^1 is linear, the second feature implies that the value function W^2 is locally linear, and the first feature implies that the slope of W^2 changes at v_1. Therefore, W^2 is piece–wise linear with a kink at v_1.

Generalizing this argument, it follows inductively that every additional time period adds an additional kink to the value function. Hence, with T periods, the value function is piece–wise linear with $T-1$ kinks. The next proposition contains the formal statement of this result.

Proposition 1. *Define the (decreasing) sequence*

$$v_t \;\equiv\; \bar{\theta}\left(\frac{\delta\bar{\theta}}{\theta_L}\right)^t, \quad t = 0, 1, 2, \ldots, \tag{15}$$

and let

$$A \equiv \frac{p\tau_H(\theta_L - \delta\bar{\theta}) + (1-p)\theta_L(\tau_L - \delta\bar{\tau})}{\bar{\theta}(\theta_L - \delta\bar{\theta})}, \qquad \sigma \equiv \frac{(1-p)\theta_L}{\bar{\theta}}. \tag{16}$$

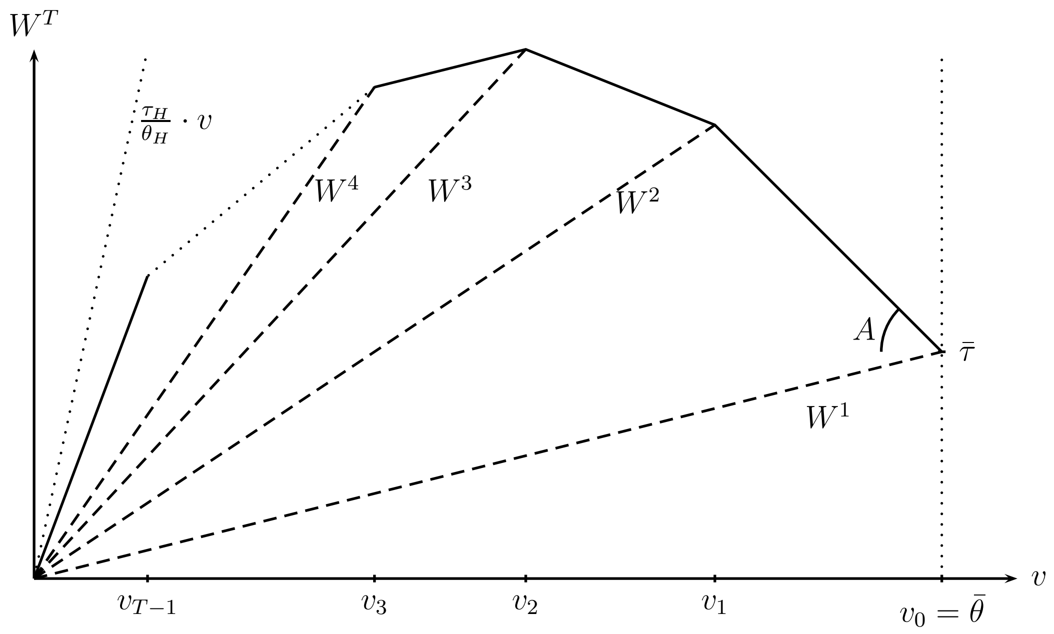


Figure 2: The principal's value function.

The principal's value function W^T is piece–wise linear with kinks at v_t, $t = 1, \ldots, T-1$. For $v \in (v_t, v_{t-1})$, $t \leq T-1$:

$$\frac{dW^T(v)}{dv} = \frac{\tau_H}{\theta_H} + \sigma^{t-1}\left(A - \frac{\tau_H}{\theta_H}\right). \tag{17}$$

For $v \in (0, v_{T-1})$:

$$\frac{dW^T(v)}{dv} = \frac{\tau_H}{\theta_H} + \sigma^{T-1}\left(\frac{\bar{\tau}}{\bar{\theta}} - \frac{\tau_H}{\theta_H}\right). \tag{18}$$

Figure 2 illustrates the principal's value function graphically. The lowest dashed line depicts the value function for time horizon $T = 1$, the next dashed line together with the appended downward sloping solid line depicts the value function for time horizon $T = 2$, and so on. With any additional period, an additional kink at v_{T-1} is added, "fanning out" the value function associated with the one period shorter time horizon. The dotted line through the origin is the upper bound on the value function derived in Lemma 3, and A is the slope of the value function at $\bar{\tau}$.

With the help of Lemma 7, we can infer the optimal mechanism from the value function. Since the value function is concave and piece–wise linear, it thus has a maximum at one of the kinks v_t, or at $v = 0$. Denote the maximizer by v^*.

We first consider two extreme cases. If $v^* = v_0 = \bar{\theta}$, then the optimal mechanism is "full delegation" (and stopping occurs immediately). If $v^* = 0$, then the optimal mechanism is "resignation" (and stopping never occurs). Note that "resignation" can only be optimal if $\bar{\tau} < 0$.

In the other cases, when $0 < v^* < \bar{\theta}$, then $v^* = v_{t^*}$ for some $t = t^* \in \{1, \dots, T-1\}$. Lemma 7 then implies that the stopping probability in period 1 is $x_H^* = v_{t^*}/\bar{\theta}$ after an H report, and the continuation value is $v_H^* = 0$. The latter means that after an initial H report, with probability $1 - x_H^*$, "resignation" occurs. If, on the other hand, the agent reports L, then, since $v_{t^*} \leq v_1$, the mechanism continues with probability 1 to period 2, and, by Lemma 7, provides the agent with a continuation value

$$v_L^* = \frac{\theta_L}{\delta\bar{\theta}} \cdot v_{t^*} = v_{t^*-1}. \tag{19}$$

The optimal way to provide the agent with that value in period 2 is now analogous to period 1: If $v_{t^*-1} = \bar{\theta}$, "full delegation" occurs. Otherwise, after an H report, the mechanism stops with probability $v_{t^*-1}/\bar{\theta}$, and "resignation" occurs with the opposite probability, and, after an L report, the mechanism continues to period 3 and provides the agent with the value $\theta_L/\delta\bar{\theta} \cdot v_{t^*-1} = v_{t^*-2}$ in period 3.

Proceeding in this fashion, if the agent reports L in each period up to and including period t^*, then the mechanism continues, without stopping, up to period $t^* + 1$ and "full delegation" occurs in period t^*+1 (in which case stopping occurs with probability 1). If the agent reports H before period t^*, the mechanism stops with some probability and with the opposite probability "resignation" occurs. We refer to t^* as the deadline of the optimal mechanism. We summarize these considerations in the next proposition.

Proposition 2. *The optimal mechanism is "resignation", or the optimal mechanism exhibits a deadline $t^* \leq T - 1$ in which case the optimal stopping probabilities are given as follows:*

- *The stopping probability after an L-report when only L's have been reported before is zero in periods $t = 1, \dots, t^*$ and is equal to 1 in period $t^* + 1$.*
- *The stopping probability after an H-report when only L's have been reported before is equal to $v_{t^*+1-t}/\bar{\theta}$ in periods $t = 1, \dots, t^*$ and is equal to 1 in period $t^* + 1$.*
- *The stopping probability after any report when one H has been reported before is zero.*

Proposition 2 describes the optimal mechanism as a direct mechanism. We now present a natural way to implement the mechanism indirectly: Unless "resignation" is optimal, the principal sets the agent the deadline t^*. Until the deadline, the agent can make at most one proposal to stop. If the agent makes a proposal, it is accepted with some probability (which depends on the time of the proposal). With the opposite probability, the proposal is rejected in which case "resignation" occurs. If the agent does not make a proposal until the deadline, then "full delegation" occurs immediately after the deadline.

In principle, the deadline could grow with the time horizon T. The next lemma shows that this, however, is not the case. Rather, there is a finite bound on the deadline, regardless of the length of the time horizon.

Lemma 8. *(i) If $\frac{\tau_H}{\theta_H} + \sigma^{T-1}(\frac{\bar{\tau}}{\bar{\theta}} - \frac{\tau_H}{\theta_H}) \leq 0$, then "resignation" is optimal.*
(ii) Otherwise, let $t^*(T)$ be the deadline of the optimal mechanism for time horizon T. Then there is $\bar{t} < \infty$ so that $t^*(T) < \bar{t}$ for all T. More precisely:

(a) If $A \geq 0$, then $t^* = 0$;

(b) If $A < 0$ and $\frac{\tau_H}{\theta_H} + \sigma^{T-2}(A - \frac{\tau_H}{\theta_H}) < 0$, then $t^* = T - 1$;

(c) Otherwise, t^* is the integer which solves

$$\frac{\tau_H}{\theta_H} + \sigma^{t^*}\left(A - \frac{\tau_H}{\theta_H}\right) > 0 > \frac{\tau_H}{\theta_H} + \sigma^{t^*-1}\left(A - \frac{\tau_H}{\theta_H}\right). \tag{20}$$

Notice that as T grows large, then because $\sigma < 1$, the cases (i) and (ii)(b) can never occur. Hence, for sufficiently large time horizon, the deadline t^* is either 0, or it is given by the (bounded) solution to (20), which shows that the deadline is bounded.

The previous considerations establish the value function and the optimal mechanism for the finite horizon case. We now use these results to establish the optimal mechanism for the infinite horizon case.

5.2 Infinite time horizon

For the finite time horizon, it is graphically evident and can be easily verified algebraically that for any v and sufficiently long time horizon, the value functions W^T do not change any more as the time horizon increases. More precisely, we have the following lemma.

Lemma 9. *For all $T \geq 2$ and $T' > T$, it holds:*

$$W^{T'}(v) = W^{T}(v) \quad \textit{for all } v \geq v_{T-1}. \tag{21}$$

Together with Lemma 2, this immediately delivers that the solution for the infinite time horizon coincides with the solution for sufficiently long finite time horizon.

Proposition 3. *It holds:*
(a) $W(v) = W^{T}(v)$ for $v \geq v_{T-1}$ and $T \geq 2$;
(b) The optimal mechanism in the infinite horizon case is the same as the optimal mechanism for the case with finite time horizon $T \geq t^*$.

Therefore, also in the infinite time horizon case, the optimal mechanism has the finite deadline t^*. Note that this is in contrast to the benchmark case with publicly observable state in which case the principal stops when state H occurs for the first time. This, however, may take longer than any given finite time horizon.

To conclude this section, we discuss two simple corollaries. From the remark after Lemma 8, it follows that in the infinite time horizon case, "resignation" is never an optimal mechanism. This is so regardless of how small the principal's value from stopping in the low state, τ_L, is. Intuitively, if τ_L is very small, the deadline will be rather large so that stopping in the low state is very unlikely and, if it occurs, it does so very late. In this sense, the adverse selection problem is never so strong that the principal would prefer to not interact with the agent at all.

In contrast, "full delegation" can be the optimal mechanism. To see this, observe that by Lemma 8, full delegation occurs if $A \geq 0$. By (2) and definition of A, it is evident that $A \geq 0$ if $\delta = 0$, or $p = 1$. Thus, we obtain the following result.

Lemma 10. *(i) "Resignation" is never optimal.*
(ii) "Full delegation" is optimal if and only if $A \geq 0$. This occurs if δ is sufficiently small, or p is sufficiently large.

6 Conclusion

Despite their prevalence in practice, optimal stopping problems with private information are only beginning to be studied in the literature. This paper is an attempt to shed light on optimal stopping rules in a principal agent framework for simple information structures when monetary transfers are infeasible. In future research, we plan to explore richer information structures and richer action spaces, look into the case with more than one agent, or study what happens when the agent has to engage in costly information acquisition so that there is moral hazard. An interesting avenue is also to relax commitment assumptions.

Appendix

Proof of Lemma 1 The claim follows directly from the discussion preceding the statement of the lemma. Q.E.D.

Proof of Lemma 2 The argument is in the main text Q.E.D.

Proof of Lemma 3 The argument for part (i) is given in the main text. Note that part (iii) follows from parts (i) and (ii) since W^T is concave and $W^T(0) = 0$. We will now prove part (ii) by induction over T.

Induction start: Observe first that since $\tau_L < \delta\bar{\tau}$ by (2), we have $\tau_L < (\delta p)/(1-\delta(1-p)) \cdot \tau_H$. Likewise, since $\theta_L > \delta\bar{\theta}$, we have $\theta_L > (\delta p)/(1-\delta(1-p)) \cdot \theta_H$. Combining the two inequalities yields

$$\frac{\tau_L}{\theta_L} < \frac{\tau_H}{\theta_H}, \tag{22}$$

which, in turn, is equivalent to

$$\frac{\bar{\tau}}{\bar{\theta}} < \frac{\tau_H}{\theta_H}. \tag{23}$$

Hence, inequality the claim in (ii) is true for W^1.

Induction step: Assume (ii) holds for $T-1 \geq 1$. By (22),

$$\begin{aligned}
W^T(v) &= p[\tau_H x_H + (1-x_H)\delta W^{T-1}(v_H)] + (1-p)[\tau_L x_L + (1-x_L)\delta W^{T-1}(v_L)] && (24)\\
&\leq p[\frac{\tau_H}{\theta_H}\theta_H x_H + (1-x_H)\delta W^{T-1}(v_H)] + (1-p)[\frac{\tau_H}{\theta_H}\theta_L x_L + (1-x_L)\delta W^{T-1}(v_L)] && (25)\\
&\leq p[\frac{\tau_H}{\theta_H}\theta_H x_H + (1-x_H)\delta\frac{\tau_H}{\theta_H}v_H] + (1-p)[\frac{\tau_H}{\theta_H}\theta_L x_L + (1-x_L)\delta\frac{\tau_H}{\theta_H}v_L] && (26)\\
&= \frac{\tau_H}{\theta_H}v, && (27)
\end{aligned}$$

where the first inequality follows by (22) and the second inequality by induction hypothesis. This establishes part (ii) for $T > 1$ and completes the proof. Q.E.D.

Proof of Lemma 4: Proof of (i): The inequality $x_H^* \geq x_L^*$ follows in the standard fashion by adding up the incentive constraints. We now prove that (IC_L) is binding. Note first that this is trivial for $x_H^* = 1$ or $v_L^* = 0$. Therefore, let $x_H^* < 1$, $v_L^* > 0$, and suppose that (IC_L) is not binding. Then marginally increase x_H^* by $\Delta x_H > 0$ and marginally change v_L^* by $\Delta v_L < 0$ so as to maintain $(\star)$:

$$p[\theta_H - \delta v_H^*]\Delta x_H + (1-p)(1-x_L^*)\delta \Delta v_L = 0. \tag{28}$$

Since the modification is marginal, it preserves (IC_L). Moreover, it clearly relaxes (IC_H). Finally, the modification changes the principal's utility by

$$\Delta W^T = p[\tau_H - \delta W^{T-1}(v_H^*)]\Delta x_H + (1-p)\delta(1-x_L^*)[W^{T-1}(v_L^* + \Delta v_L) - W^{T-1}(v_L^*)]. \tag{29}$$

Multiplying (28) by τ_H/θ_H and subtracting it from ΔW^T yields

$$\begin{aligned} \Delta W^T &= p[\tau_H - \tau_H/\theta_H \cdot \theta_H]\Delta x_H + p\delta[\tau_H/\theta_H \cdot v_H^* - W^{T-1}(v_H^*)]\Delta x_H + \qquad (30) \\ &\quad +(1-p)(1-x_L^*)\delta[W^{T-1}(v_L^* + \Delta v_L) - W^{T-1}(v_L^*) - \tau_H/\theta_H \cdot \Delta v_L] \\ &= p\delta[\tau_H/\theta_H \cdot v_H^* - W^{T-1}(v_H^*)]\Delta x_H + \qquad (31) \\ &\quad (1-p)(1-x_L^*)\delta\Delta v_L \left[\frac{W^{T-1}(v_L^* + \Delta v_L) - W^{T-1}(v_L^*)}{\Delta v_L} - \tau_H/\theta_H\right]. \end{aligned}$$

The first term on the right hand side is positive by part (ii) of Lemma 3, and the second term is positive because $\Delta v_L < 0$ and because the square bracket is negative by part (iii) of Lemma 3. Thus, the modified mechanism improves the principal's utility, contradicting the optimality of the original mechanism. This shows (i).

The proof of (ii) is standard and therefore omitted.

Proof of Lemma 5: "$\Rightarrow$": $(IC_L^=)$ and $(\tilde{\star})$ imply

$$x_H = \frac{v - \delta v_H}{\bar{\theta} - \delta v_H}, \quad x_L = \frac{(\theta_L - \bar{\theta})x_H + v - \delta v_L}{\theta_L - \delta v_L}. \tag{32}$$

We first show that $\underline{x}_H \leq x_H \leq \bar{x}_H$. Indeed, observe that x_H is decreasing in v_H since $v \leq \bar{\theta}$.[7] Since $v_H \in [0, \bar{\theta}]$, it follows that x_H is in the range

$$\frac{v - \delta\bar{\theta}}{\bar{\theta} - \delta\bar{\theta}} \leq x_H \leq \frac{v}{\bar{\theta}} = \bar{x}_H. \tag{33}$$

[7] $\frac{d}{dv}\frac{A-\delta v}{B-\delta v} = \frac{\delta(A-B)}{(B-\delta v)^2} \geq 0 \Leftrightarrow A \geq B.$

Moreover, since $x_H \geq 0$ by condition (M), we have that

$$\underline{x}_H \;\equiv\; \max\left\{0, \frac{v - \delta\bar{\theta}}{\bar{\theta} - \delta\bar{\theta}}\right\} \leq \bar{x}_H, \tag{34}$$

as desired. Next, we show that $\max\{0, \xi_L(x_H)\} \leq x_L \leq x_H$. Fix x_H. We show below that for all v:

$$(\theta_L - \bar{\theta})x_H + v \leq \theta_L. \tag{35}$$

This implies that x_L as defined in (32) is decreasing in v_H. Moreover, by condition (M), $0 \leq x_L \leq x_H$. Taken together, this restricts the range of x_L to

$$\max\left\{0, \frac{(\theta_L - \bar{\theta})x_H + v - \delta\bar{\theta}}{\theta_L - \delta\bar{\theta}}\right\} \leq x_L \leq x_H. \tag{36}$$

Given the definition of ξ_L, this is what we wanted to show. It remains to show (35). Consider first the case that $v < \delta\bar{\theta}$. In this case $\underline{x}_H = 0$, and since $\theta_L - \bar{\theta} < 0$, we can deduce:

$$(\theta_L - \bar{\theta})x_H + v \leq (\theta_L - \bar{\theta})\underline{x}_H + v = v \leq \delta\bar{\theta} \leq \theta_L, \tag{37}$$

where the final inequality follows by assumption, establishing (35) for $v < \delta\bar{\theta}$. Consider next the case that $v \geq \delta\bar{\theta}$. In this case $\underline{x}_H = \frac{v-\delta\bar{\theta}}{\bar{\theta}-\delta\bar{\theta}} > 0$, and since $\theta_L - \bar{\theta} < 0$, we have:

$$\begin{aligned}(\theta_L - \bar{\theta})x_H + v \;&\leq\; (\theta_L - \theta)\underline{x}_H + v && (38)\\ &=\; (\theta_L - \bar{\theta})\frac{v - \delta\bar{\theta}}{\bar{\theta} - \delta\bar{\theta}} + v. && (39)\end{aligned}$$

The right hand side is smaller than θ_L if and only if

$$\begin{aligned} & \theta_L v - \theta_L\delta\bar{\theta} - \bar{\theta}v + \delta\bar{\theta}^2 + v\bar{\theta} - v\delta\bar{\theta} \leq \theta_L\bar{\theta} - \theta_L\delta\bar{\theta} && (40)\\ \Leftrightarrow\quad & (v - \bar{\theta})(\theta_L - \delta\bar{\theta}) \leq 0, && (41)\end{aligned}$$

which is always true. Hence, as desired, (35) is shown, and this completes the proof of the "$\Rightarrow$"–part.

"$\Leftarrow$": Let $x \in X$. Define v_H and v_L by (10). It is easy to see that this implies $(IC_L^{=})$ and $(\tilde{\star})$. To see that also condition (M) is satisfied, observe first that by definition of X, we clearly have that $0 \leq x_L \leq x_H \leq 1$. Second, to see that $v_H, v_L \in [0, \bar{\theta}]$, consider first v_H. Observe that $v_H(x_H)$ as defined in (10) is decreasing in x_H. Hence, given the range of x_H, v_H is bounded from below by

$$v_H(\bar{x}_H) = \frac{1}{\delta}\frac{v - v}{\frac{\bar{\theta}-v}{\bar{\theta}}} = 0,$$

and v_H is bounded from above by $v_H(\underline{x}_H)$, which, for $v < \delta\bar{\theta}$ is equal to

$$v_H(0) = \frac{v}{\delta} < \frac{\delta\bar{\theta}}{\delta} = \bar{\theta},$$

and, for $v \geq \delta\bar{\theta}$ is equal to

$$v_H(\underline{x}_H) = \frac{1}{\delta}\frac{\frac{v\bar{\theta}-v\delta\bar{\theta}-\bar{\theta}v+\delta\bar{\theta}^2}{\bar{\theta}-\delta\bar{\theta}}}{\frac{\bar{\theta}-\delta\bar{\theta}-v+\delta\bar{\theta}}{\bar{\theta}-\delta\bar{\theta}}} = \frac{\delta\bar{\theta}(\bar{\theta}-v)}{\bar{\theta}-v} = \bar{\theta}.$$

Similar computations show that that v_L is in between 0 and $\bar{\theta}$, and this completes the proof of the "$\Leftarrow$"–part. Q.E.D.

Proof of Lemma 6 (i) The argument is in the main text. To see part (ii), let

$$\begin{aligned} \bar{W}^T(x_H, x_L; v) \;\; &\equiv \;\; p[\tau_H x_H + (1-x_H)\delta W^{T-1}(v_H(x_H))] && (42)\\ &\quad +(1-p)[\tau_L x_L + (1-x_L)\delta W^{T-1}(v_L(x_H, x_L))] && (43)\end{aligned}$$

be the principal's objective after inserting (10) in the value function W^{T-1}. To show the claim, we show that $\bar{W}^T$ is decreasing in x_L. Let

$$\frac{dW^T(\bar{\theta})}{dv^-} = \lim_{v\uparrow\bar{\theta}} \frac{W^T(\bar{\theta}) - W^T(v)}{\bar{\theta}-v} \tag{44}$$

be the left derivative of W^T at the point $\bar{\theta}$ whenever the limit exists. Moreover, let

$$A \equiv \frac{p\tau_H(\theta_L - \delta\bar{\theta}) + (1-p)\theta_L(\tau_L - \delta\bar{\tau})}{\bar{\theta}(\theta_L - \delta\bar{\theta})}. \tag{45}$$

We prove the claim in two steps. We first show:

$$\bar{W}^2(x_H, x_L) \text{ is decreasing in } x_L. \tag{46}$$

We then show for all $T \geq 2$:

$$\bar{W}^T \text{ is decreasing in } x_L \quad \Rightarrow \quad \frac{dW^T(\bar{\theta})}{dv^-} = A \quad \Rightarrow \quad \bar{W}^{T+1} \text{ is decreasing in } x_L. \tag{47}$$

To see (46), recall that $W^1(v) = \bar{\tau}/\bar{\theta}\cdot v$. Hence,

$$\begin{aligned} \bar{W}^2(x_H, x_L) \;\; &= \;\; p[\tau_H x_H + (1-x_H)\delta\cdot\bar{\tau}/\bar{\theta}\cdot v_H(x_H)] && (48)\\ &\quad +(1-p)[\tau_L x_L + (1-x_L)\delta\cdot\bar{\tau}/\bar{\theta}\cdot v_L(x_H, x_L)]. && (49)\end{aligned}$$

After plugging in (10):

$$
\begin{aligned}
\bar{W}^2(x_H, x_L, v) &= p[\tau_H x_H + \bar{\tau}/\bar{\theta} \cdot (v - \bar{\theta} x_H)] + && (50)\\
&\quad +(1-p)[\tau_L x_L + \bar{\tau}/\bar{\theta} \cdot ((\theta_L - \bar{\theta})x_H - \theta_L x_L + v)] && (51)\\
&= (1-p)[\tau_L - \bar{\tau}/\bar{\theta} \cdot \theta_L] x_L + K, && (52)
\end{aligned}
$$

where K is a constant that does not depend on x_L. Since $\tau_L < \bar{\tau}/\bar{\theta} \cdot \theta_L$ by assumption[8], this implies (46).

We next prove the first implication of (47). If $\bar{W}^T$ is decreasing in x_L, and since $\bar{W}^T$ is increasing in x_H by part (i), it is optimal to choose x_L as small and x_H as large as possible. In particular, for $v \geq \delta\bar{\theta}^2/\theta_L$, Lemma 5 therefore implies that the optimal mechanism is

$$
x_H^* = \frac{v}{\bar{\theta}}, \qquad x_L^* = \frac{\theta_L/\bar{\theta} \cdot v - \delta\bar{\theta}}{\theta_L - \delta\bar{\theta}}, \qquad v_H^* = 0, \qquad v_L^* = \bar{\theta}. \tag{53}
$$

Inserting this in $\bar{W}^T$ delivers:

$$
\begin{aligned}
W^T(v) = \bar{W}^T(x_H^*, x_L^*, v) &= p\tau_H x_H^* + (1-p)[\tau_L x_L^* + \delta(1 - x_L^*) W^{T-1}(\bar{\theta})] && (54)\\
&= p\tau_H/\bar{\theta} \cdot v + (1-p)[\tau_L \frac{\theta_L/\bar{\theta} \cdot v - \delta\bar{\theta}}{\theta_L - \delta\bar{\theta}} + \delta \frac{\theta_L - \theta_L/\bar{\theta} \cdot v}{\theta_L - \delta\bar{\theta}} \bar{\tau}] && (55)\\
&= \frac{1}{\bar{\theta}(\theta_L - \delta\bar{\theta})}[p\tau_H(\theta_L - \delta\bar{\theta}) + (1-p)\theta_L(\tau_L - \delta\bar{\tau})]v + k && (56)\\
&= A \cdot v + k, && (57)
\end{aligned}
$$

where k is a constant that does not depend on v. Since this is true for all for $v \geq \delta\bar{\theta}^2/\theta_L$, we obtain:

$$
\frac{dW^T(\bar{\theta})}{dv^-} = \lim_{v \uparrow \bar{\theta}} \frac{W^T(\bar{\theta}) - W^T(v)}{\bar{\theta} - v} = \lim_{v \uparrow \bar{\theta}} A = A, \tag{58}
$$

and this is what we wanted to show.

Finally, we prove the right implication of of (47). Because W^T is concave, it is differentiable almost everywhere. Since, moreover, v_L is differentiable in x_L, it follows that $\bar{W}^{T+1}$ is differentiable in x_L almost everywhere. To show that $\bar{W}^{T+1}$ is decreasing in x_L, it is therefore sufficient to show that

$$
\frac{\partial \bar{W}^{T+1}(x_L)}{\partial x_L} \leq 0 \tag{59}
$$

[8] $\tau_L < \delta\bar{\tau}$ and $\theta_L > \delta\bar{\theta}$ implies $\tau_L/\theta_L > \delta\bar{\tau}/(\delta\bar{\theta}) = \bar{\tau}/\bar{\theta}$.

at all points x_L where the derivative exists. To see (59), note first that by (10), v_H does not depend on x_L. Hence, taking the derivative of $(IC_L^=)$ with respect to x_L yields that

$$\theta_L - \delta v_L + (1-x_L)\delta \frac{\partial v_L}{\partial x_L} = 0. \tag{60}$$

Hence,

$$\begin{aligned} \frac{\partial \bar{W}^{T+1}(x_L)}{\partial x_L} &= (1-p)\left[\tau_L - \delta W^T(v_L) + (1-x_L)\delta \frac{dW^T(v_L)}{dv}\frac{\partial v_L}{\partial x_L}\right] && (61)\\ &= (1-p)\left[\tau_L - \delta W^T(v_L) + \frac{dW^T(v_L)}{dv}(\delta v_L - \theta_L)\right]. && (62)\end{aligned}$$

Since W^T is concave, we have that

$$W^T(\bar{\theta}) - W^T(v_L) \leq \frac{dW^T(v_L)}{dv}(\bar{\theta} - v_L), \tag{63}$$

and, moreover, that

$$\frac{dW^T(v_L)}{dv} \geq \frac{dW^T(\bar{\theta})}{dv^-} = A, \tag{64}$$

where the equality is true by hypothesis. Using these inequalities in turn, we get:

$$\begin{aligned} \frac{\partial \bar{W}^{T+1}(x_L)}{\partial x_L} &\leq (1-p)\left[\tau_L - \delta W^T(\bar{\theta}) + \frac{dW^T(v_L)}{dv}(\delta\bar{\theta} - \theta_L)\right] && (65)\\ &\leq (1-p)\left[\tau_L - \delta W^T(\bar{\theta}) + A(\delta\bar{\theta} - \theta_L)\right], && (66)\end{aligned}$$

where the second inequality follows since $(\delta\bar{\theta} - \theta_L) < 0$. Now note that $W^T(\bar{\theta}) = \bar{\tau}$ and that $A \geq (\tau_L - \delta\bar{\tau})/(\theta_L - \delta\bar{\theta})$. Hence, the right hand side of the previous inequality is smaller than

$$(1-p)\left[\tau_L - \delta\bar{\tau} + \frac{\tau_L - \delta\bar{\tau}}{\theta_L - \delta\bar{\theta}}(\delta\bar{\theta} - \theta_L)\right] = 0. \tag{67}$$

This shows (59) and completes the proof. Q.E.D.

Proof of Lemma 7 The claim follows from the discussion preceding the statement of the lemma. Q.E.D.

Proof of Proposition 1 The proof is by induction over T.

Induction start: For $T = 1$, observe that $v_{T-1} = v_0 = \bar{\theta}$. Hence, we only have to show that for all $v \in (0, \bar{\theta})$, we have $dW^1/dv = \bar{\tau}/\bar{\theta}$. But this is immediate from the definition of W^1 in (6).

Induction step: Let the claim be true for $T-1$. We distinguish two cases. For $v > v_1$, we want to show that the derivative of the value function is equal to A. Indeed, by Lemma 7 we have $x_H^* = v/\bar{\theta}$, $v_H^* = 0$, $x_L^* = (\theta_L/\bar{\theta} \cdot v - \delta\bar{\theta})/(\theta_L - \delta\bar{\theta})$, and $v_L^* = \bar{\theta}$. Since $W^{T-1}(v_L^*) = W^{T-1}(\bar{\theta}) = \bar{\tau}$, we get

$$W^T(v) \;=\; p\tau_H \frac{v}{\bar{\theta}} + (1-p)\left[\tau_L \frac{\theta_L/\bar{\theta} \cdot v - \delta\bar{\theta}}{\theta_L - \delta\bar{\theta}} + \frac{\theta_L - \theta_L/\bar{\theta} \cdot v}{\theta_L - \delta\bar{\theta}}\delta\bar{\tau}\right]. \tag{68}$$

This is linear in v, and the derivative is

$$\frac{dW^T(v)}{dv} = p\frac{\tau_H}{\bar{\theta}} + (1-p)\left[\tau_L \frac{\theta_L/\bar{\theta}}{\theta_L - \delta\bar{\theta}} - \frac{\theta_L/\bar{\theta}}{\theta_L - \delta\bar{\theta}}\delta\bar{\tau}\right] = p\frac{\tau_H}{\bar{\theta}} + (1-p)\frac{\theta_L(\tau_L - \delta\bar{\tau})}{\bar{\theta}(\theta_L - \delta\bar{\theta})} = A, \tag{69}$$

which is is what we wanted to show.

Next, consider values $v \leq v_1$. By Lemma 7, $x_H^* = v/\bar{\theta}$, $v_H^* = 0$, $x_L^* = 0$, and $v_L^* = \theta_L/\delta\bar{\theta} \cdot v$. Hence,

$$W^T(v) \;=\; p\tau_H \frac{v}{\bar{\theta}} + (1-p)\delta W^{T-1}\left(\frac{\theta_L}{\delta\bar{\theta}}v\right). \tag{70}$$

Now if $v \in (v_t, v_{t-1})$ for $2 \leq t \leq T-1$, then $\theta_L/\delta\bar{\theta} \cdot v \in (v_{t-1}, v_{t-2})$ by definition of v_t. By induction hypothesis, the slope of W^{T-1} in this range is $\frac{\tau_H}{\theta_H} + \sigma^{t-2}\left(A - \frac{\tau_H}{\theta_H}\right)$, and thus with $\sigma = (1-p)\theta_L/\bar{\theta}$, we obtain:

$$\begin{aligned}
\frac{dW^T(v)}{dv} &= p\frac{\tau_H}{\bar{\theta}} + (1-p)\delta\left(\frac{\tau_H}{\theta_H} + \sigma^{t-2}\left(A - \frac{\tau_H}{\theta_H}\right)\right)\frac{\theta_L}{\delta\bar{\theta}} && (71)\\
&= p\frac{\tau_H}{\bar{\theta}} + (1-p)\frac{\theta_L}{\bar{\theta}}\frac{\tau_H}{\theta_H} + \sigma^{t-1}\left(A - \frac{\tau_H}{\theta_H}\right) && (72)\\
&= \frac{\tau_H}{\theta_H} + \sigma^{t-1}\left(A - \frac{\tau_H}{\theta_H}\right), && (73)
\end{aligned}$$

as desired.

Finally, if $v \in (0, v_{T-1})$, then $\theta_L/\delta\bar{\theta} \cdot v \in (v_{T-1}, v_{T-2})$ by definition of v_t. By induction hypothesis, the slope of W^{T-1} in this range is $\frac{\tau_H}{\theta_H} + \sigma^{T-2}\left(\frac{\bar{\tau}}{\bar{\theta}} - \frac{\tau_H}{\theta_H}\right)$, and thus with $\sigma = (1-p)\theta_L/\bar{\theta}$, we otbain:

$$\begin{aligned}
\frac{dW^T(v)}{dv} &= p\frac{\tau_H}{\bar{\theta}} + (1-p)\delta\left(\frac{\tau_H}{\theta_H} + \sigma^{T-2}\left(\frac{\bar{\tau}}{\bar{\theta}} - \frac{\tau_H}{\theta_H}\right)\right)\frac{\theta_L}{\delta\bar{\theta}} && (74)\\
&= p\frac{\tau_H}{\bar{\theta}} + (1-p)\frac{\theta_L}{\bar{\theta}}\frac{\tau_H}{\theta_H} + \sigma^{T-1}\left(\frac{\bar{\tau}}{\bar{\theta}} - \frac{\tau_H}{\theta_H}\right) && (75)\\
&= \frac{\tau_H}{\theta_H} + \sigma^{T-1}\left(\frac{\bar{\tau}}{\bar{\theta}} - \frac{\tau_H}{\theta_H}\right). && (76)
\end{aligned}$$

And this completes the proof.

Proof of Proposition 2 The claim follows from the discussion preceding the statement of the lemma. Q.E.D.

Proof of Lemma 8 Since W^T is piece–wise linear and concave, W^T is globally increasing if the slope in the segment $(v_1, \bar{\theta})$, given by A, is positive, in which case $t^* = 0$. Otherwise, v_{t^*} is the (unique) point where the slope of W^T changes from positive into negative. Using Proposition 1, the conditions in the lemma characterize when this is the case. Q.E.D.

Proofs of Lemma 9, Proposition 3, and Lemma 10 The arguments are given in the main text. Q.E.D.

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